Partial Differential Equations - Midterm exam

You have 2 hours to answer the following questions, and 20 minutes of scan time. All questions are worth 30 points. Everyone receives 10 points for turning in their exam. This exam is open book and open note, but not open internet. Justify all work with appropriate theorems. Late answers will not be accepted.

(1) Prove the solution to the following initial boundary value problem:

$$\begin{aligned}
&\partial_t u(t,x) = \Delta u(t,x) \quad \text{in} \quad \Omega \\
&\frac{\partial u}{\partial \nu}(t,x) = g(t,x) \quad \text{on} \quad \partial\Omega \\
&u(0,x) = f(x) \quad \text{in} \quad \Omega
\end{aligned}$$

is unique for positive times t and $x \in \Omega$. You may assume that $f \in C^2(\overline{\Omega})$, and $g \in C^2(\mathbb{R}_+ \times \partial\Omega)$. The set Ω is a bounded open domain with ν being the outward pointing normal vector to $\partial\Omega$.

(2) a) Use the method of separation of variables to solve the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2}(x,t) = \frac{\partial^2 \phi}{\partial x^2}(x,t)$$

for the function with boundary conditions

$$\phi(0,t) = 0 \quad \text{and} \quad \phi(\pi,t) = 0$$

and the initial conditions

$$\phi(x,0) = \sin^3 x \quad \text{and} \quad \frac{\partial \phi}{\partial t}(x,0) = 0$$

b) Solve instead with boundary conditions

$$\phi(0,t) = 0 \quad \text{and} \quad \phi(\pi,t) = \pi$$

and the initial conditions

$$\phi(x,0) = \sin^3 x + x \quad \text{and} \quad \frac{\partial \phi}{\partial t}(x,0) = 0.$$

c) The general solution is of the form:

$$f(x+t) + g(x-t)$$

Rewrite the solution in part b) in this form. Determine f and g.

(3) Solve the equation

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) - x\frac{\partial u}{\partial x}(t,x) = 0 \\
&u(0,x) = \frac{1}{x^2+1}
\end{aligned}$$

Find

$$\lim_{t\to\infty} u(t,x)$$

and carefully justify the limit.